Figure Captions

Fig.1. Region of absolute convergence of the solutions $J_1[1]$, J_3 and J_4.

Fig.2. Region of absolute convergence of the H_2 hypergeometric function.

Fig.3. Region of absolute convergence of $J_2[1]$, J_5 and J_6. The curve that separates J_2 of the others is a branch cut, see eq.(46).

Fig.4. Region of absolute convergence of the J_8. The solution J_7 is symmetric to it in $s \leftrightarrow t$. They are finite and hold in the relativistic regime of forward scattering in the t and s-channel respectively.

Fig.5. The general 2-particle $\rightarrow$ 2-particle Feynman graph that has leading singularity(ies) on the physical sheet[17].

NEGATIVE DIMENSIONAL INTEGRATION FOR MASSIVE FOUR POINT FUNCTIONS–II: NEW SOLUTIONS

Alfredo T. Suzuki and Alexandre G. M. Schmidt [1]

Instituto de Física Teórica – Universidade Estadual Paulista, R.Pamplona, 145, São Paulo SP, CEP 01405-900, Brazil

Abstract

In this sequel calculation of the one-loop Feynman integral pertaining to a massive box diagram contributing to the photon-photon scattering amplitude in quantum electrodynamics, we present the six solutions as yet unknown in the literature. These **six new solutions** arise quite naturally in the context of negative dimensional integration approach, revealing a promising technique to handle Feynman integrals.

Key words: Feynman Integrals, Massive Feynman Box Diagram, Negative Dimensional Integration Method.
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[1] E-mail:suzuki@power.ift.unesp.br, schmidt@power.ift.unesp.br

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1 Introduction.

In our previous paper[1] we calculated the Feynman integral pertaining to the photon-photon scattering amplitude in Quantum Electrodynamics (QED) using a technique we call negative dimensional integration method (NDIM)[2–5]. One of the outstanding features of NDIM is that the complexities of performing D-dimensional momentum integrals are transferred to the easier task of solving systems of linear algebraic equations. In [1] NDIM has allowed us to recover very easily the two known hypergeometric series representations for the pertinent Feynman integral. Another outstanding feature of NDIM — and this is in our opinion its greatest potential — is that it gives simultaneously six new results for the integral in question in a very straightforward manner. The aim of the present paper is to consider these new results. Each of them is valid in certain regions of external momenta and are related to the others by analytic continuation, either directly or indirectly.

Compared to the traditional methods of working out Feynman integrals, NDIM is by far simpler and more straightforward. The difficulties of calculating parametric integrals — if one is handling a Feynman integral in the standard way[6] — or solving contour integrals — if one is using the Mellin-Barnes' integral representation for massive propagators[7–9] — are with finesse embedded in an easier problem of solving systems of linear algebraic equations.

The Mellin-Barnes' integral technique can provide us with only two types of results: either the resulting function will depend on, say, p^2/m^2, or its inverse, m^2/p^2, depending on whether one chooses to close the contour to the left or to the right in the complex plane. The resulting functions are in general hypergeometric ones. However, if we have more than one external momentum (as in our present case) and/or internal mass, there are several other combinations of dimensionless variables that define distinct regions of external momenta. It is then clear that the Mellin-Barnes' technique will be limited to give simultaneously only two among those several possibilities. Not so with NDIM, where many of the possibilities are accounted for at the same time. Of course, in both cases, one can always construct other power series representations by means of suitable analytic continuations as long as the formula for such extensions be known. NDIM has then clear advantages over other techniques in this sense.

Hypergeometric functions of one and two variables have lots of well-known analytic continuation formulas but as the number of variables increases — as far as we know — the fewer the known relations[10] are. On the other hand, since NDIM provides us with very many simultaneous results, which in principle must be connected by analytic continuation, we come to the realization that it is not only a very powerful technique to work out Feynman integrals

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but an elegant approach to check on analytic continuation properties of the resulting functions as well. Consider our present case: Altogether we have eigth distinct solutions for the Feynman integral for the photon-photon scattering, two of which have already been considered in our previous paper and six new ones, which are connected with each other by suitable analytic continuation formulas.

One could rightfully ask: Why do we want so many distinct results at the same time? We give some good arguments for this. Firstly, if we have only one[2] or at most two[3] results in distinct regions of the external momenta, all the other regions must be worked out all the way through the analytic continuation formulas, which is not always an easy task to perform and certainly is very much time-consuming. Secondly, the important special case of forward scattering in the relativistic regime cannot be dealt with if one has only the two known hypergeometric series representations for the Feynman integral relative to the photon-photon scattering. These series are unsuitable for handling this special case because of the very nature of their variables. The same reasoning applies to the backward scattering. Thirdly, our results are expressed in a compact form that can be transformed — by an appropriate integral representation — into the more cumbersome standard form in terms of dilogarithms, if one wants to do so. Fourthly, we can identify the branch points and singularities of Feynman integrals directly from their hypergeometric series representations. Fifthly and lastly, since any two distinct solutions are related by analytic continuation, NDIM is an elegant and economical way of obtaining analytic continuation formulas among hypergeometric series (see Section 3).

The outline for this paper is as follows: In Section 2 we write down the new results for the Feynman integral in D-dimensions and arbitrary powers of propagators much the same as in dimensional regularization[11]. In all of the resulting expressions we have singularities (poles) either when we want to take the physical limit of $D = 4$ or in the limits of unity powers for propagators, so that Section 3 is devoted to analyse the special cases we are interested in by introducing suitable regularization parameters. Four of the solutions remain divergent even after the regularization procedure and deserve our special analysis and discussion. We also examine the convergence regions for the new results and draw diagrams for all of them. Finally in the last section, Section 4, we make our concluding remarks and present some further challenges and applications for NDIM.

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[2] via parametric integration
[3] via Mellin-Barnes' integration

2 New Results from NDIM.

In our previous paper[1] we studied the integral,

$$J(i, j, k, l; m) = \int d^D q \left(q^2 - m^2 \right)^i \left[(q - p)^2 - m^2 \right]^j \left[(q - k_1)^2 - m^2 \right]^k$$
$$\times \left[(q - k_2)^2 - m^2 \right]^l, \tag{1}$$

whose counterpart in positive D is

$$K(i, j, k, l; m) = \int \frac{d^D q}{(q^2 - m^2)^i \left[(q - p)^2 - m^2 \right]^j \left[(q - k_1)^2 - m^2 \right]^k}$$
$$\times \frac{1}{\left[(q - k_2)^2 - m^2 \right]^l},$$

which is relevant to the photon-photon scattering in QED. In particular, we are interested in the case $J(-1, -1, -1, -1; m) \equiv K(1, 1, 1, 1; m)$.

Hereafter, unless otherwise noted, we closely follow the notation of [1]. So, let $\sigma = i + j + k + l + \frac{1}{2}D$, where D is the space-time dimension and let the Pochhammer symbol be

$$(a|k) \equiv (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}.$$

Here we remind ourselves that the results of (1) come from the solutions of a system of linear algebraic equations and that for this particular integral the system is such that has twenty-one solutions altogether, six of which are trivial ones and fifteen of which give double series. In [1] we listed five of them which combined appropriately among themselves led to the two well-known hyper-geometric series representations[8] of the Feynman integral. Below we give the remaining ten solutions for the system, which combined appropriately among themselves, yield the six new solutions for the pertinent Feynman integral. In the following we use s and t for the usual Mandelstam variables.

$$I_6 = f_6 \, \mathcal{S}_1 \left(\alpha_6, \alpha_6', \beta_6, \beta_6', \theta_6; \gamma_6, \theta_6' \left| \frac{-t}{s}, \frac{4m^2}{s} \right. \right) \tag{2}$$

$$I_7 = I_6(i \leftrightarrow k, j \leftrightarrow l \mid s \leftrightarrow t) \tag{3}$$

$$I_8 = f_8 \, \mathcal{S}_2 \left(\alpha_8, \beta_8, \gamma_8, \delta_8, \phi_8; \rho_8, \phi_8' \left| \frac{-t}{s}, \frac{-4m^2}{t} \right. \right) \tag{4}$$

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$$I_9 = I_8(k \leftrightarrow l) \tag{5}$$

$$I_{10} = f_{10}\, \mathcal{S}_2\left(\alpha_{10}, \beta_{10}, \gamma_{10}, \delta_{10}, \phi_{10}; \rho_{10}, \phi'_{10} \left| \frac{-s}{t}, \frac{-4m^2}{s} \right.\right) \tag{6}$$

$$I_{11} = I_{10}(i \leftrightarrow j) \tag{7}$$

$$I_{12} = f_{12}\, \mathcal{S}_2\left(\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12}, \phi_{12}; \rho_{12}, \phi'_{12} \left| \frac{4m^2}{s}, \frac{-t}{4m^2} \right.\right) \tag{8}$$

$$I_{13} = I_{12}(k \leftrightarrow l) \tag{9}$$

$$I_{14} = f_{14}\, \mathcal{S}_2\left(\alpha_{14}, \beta_{14}, \gamma_{14}, \delta_{14}, \phi_{14}; \rho_{14}, \phi'_{14} \left| \frac{4m^2}{t}, \frac{-s}{4m^2} \right.\right) \tag{10}$$

$$I_{15} = I_{14}(i \leftrightarrow j) \,, \tag{11}$$

where we have defined the following two functions

$$
\begin{aligned}
\mathcal{S}_1(\alpha, \alpha', \beta, \beta', \theta; \gamma, \theta'|z_1, z_2) = \sum_{\mu,\nu=0}^{\infty} \frac{z_1^\mu z_2^\nu}{\mu!\nu!} \frac{(\alpha|\mu)(\alpha'|\nu)(\beta|\mu)(\beta'|\nu)}{(\gamma|\mu+\nu)} \\
\times \frac{(\theta|\mu+\nu)}{(\theta'|\mu+\nu)} \,,
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
\mathcal{S}_2(\alpha, \beta, \gamma, \delta, \phi; \rho, \phi'|z_1, z_2) = \sum_{\mu,\nu=0}^{\infty} \frac{z_1^\mu z_2^\nu}{\mu!\nu!} \frac{(\alpha|\mu-\nu)(\beta|\mu)(\gamma|\nu)(\delta|\nu)}{(\rho|\mu)} \\
\times \frac{(\phi|\nu-\mu)}{(\phi'|\nu-\mu)} \,,
\end{aligned} \tag{13}
$$

with

$$
\begin{aligned}
f_6 &= (-\pi)^{D/2} \left(\frac{s}{4}\right)^\sigma \frac{(-i|\sigma)(-j|\sigma)}{(\frac{1}{2} - \frac{1}{2}\sigma|\sigma + \frac{1}{4}D)(-\frac{1}{2}\sigma|\sigma + \frac{1}{4}D)} \,, \\
f_8 &= (-\pi)^{D/2}\, s^l\, t^{\sigma-l} \frac{(-k|-i-j-\frac{1}{2}D)(-i|i-k+l)(-j|\sigma-l)}{(-i-l+\sigma|i+l+\frac{1}{2}D)} \,, \\
f_{10} &= (-\pi)^{D/2}\, s^j\, t^{\sigma-j} \frac{(-l|-i-j-\frac{1}{2}D)(-i|i+k-l)(-j|\sigma-k)}{(-i-k+\sigma|i+k+\frac{1}{2}D)} \,, \\
f_{12} &= (-\pi)^{D/2}\, s^l\, (-m^2)^{\sigma-l} \frac{(-k|l)(\sigma + \frac{1}{2}D|l - 2\sigma - \frac{1}{2}D)}{(\sigma + \frac{1}{2}D|2l - 2\sigma)} \,, \\
f_{14} &= (-\pi)^{D/2}\, s^j\, (-m^2)^{\sigma-j} \frac{(-i|j)(\sigma + \frac{1}{2}D|j - 2\sigma - \frac{1}{2}D)}{(\sigma + \frac{1}{2}D|2j - 2\sigma)} \,,
\end{aligned}
$$

with the following parameters for the function $\mathcal{S}_1$:

$$\alpha_6 = -k,$$
$$\alpha_6' = \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D,$$
$$\beta_6 = -l,$$
$$\beta_6' = 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D,$$
$$\theta_6 = -\sigma,$$
$$\gamma_6 = 1 + i - \sigma,$$
$$\theta_6' = 1 + j - \sigma,$$

and the parameters for the function $\mathcal{S}_2$:

$$\alpha_8 = j + k + \tfrac{1}{2}D, \qquad\qquad \alpha_{10} = i + k + \tfrac{1}{2}D,$$
$$\beta_8 = -l, \qquad\qquad \beta_{10} = -j,$$
$$\gamma_8 = 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D, \qquad\qquad \gamma_{10} = 1 - \tfrac{1}{2}\sigma - \tfrac{1}{4}D,$$
$$\delta_8 = \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D, \qquad\qquad \delta_{10} = \tfrac{1}{2} - \tfrac{1}{2}\sigma - \tfrac{1}{4}D,$$
$$\phi_8 = -i - j - k - \tfrac{1}{2}D, \qquad\qquad \phi_{10} = -i - k - l - \tfrac{1}{2}D,$$
$$\rho_8 = 1 + k - l, \qquad\qquad \rho_{10} = 1 + i - j,$$
$$\phi_8' = 1 - i - k - \tfrac{1}{2}D, \qquad\qquad \phi_{10}' = 1 - i - l - \tfrac{1}{2}D,$$

$$\alpha_{12} = \tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k - \tfrac{1}{2}l, \qquad\qquad \alpha_{14} = \tfrac{1}{2} + \tfrac{1}{2}i - \tfrac{1}{2}j + \tfrac{1}{2}k + \tfrac{1}{2}l,$$
$$\beta_{12} = -l, \qquad\qquad \beta_{14} = -j,$$
$$\gamma_{12} = -i, \qquad\qquad \gamma_{14} = -k,$$
$$\delta_{12} = -j, \qquad\qquad \delta_{14} = -l,$$
$$\phi_{12} = -i - j - k - \tfrac{1}{2}D, \qquad\qquad \phi_{14} = -i - k - l - \tfrac{1}{2}D,$$
$$\rho_{12} = 1 + k - l, \qquad\qquad \rho_{14} = 1 + i - j,$$
$$\phi_{12}' = -\tfrac{1}{2}i - \tfrac{1}{2}j - \tfrac{1}{2}k + \tfrac{1}{2}l, \qquad\qquad \phi_{14}' = -\tfrac{1}{2}i + \tfrac{1}{2}j - \tfrac{1}{2}k - \tfrac{1}{2}l .$$

Observe that when the parameters θ and θ' in $\mathcal{S}_1$ are equal, then our defined function $\mathcal{S}_1$ becomes the known Appel's hypergeometric function F_3, whereas when the parameters ϕ and ϕ' in $\mathcal{S}_2$ are equal, our defined function $\mathcal{S}_2$ reduces to the known Appel's hypergeometric function H_2 (see Section 3).

Looking carefully at these one can verify without difficulty that there are symmetry relations among them. For example, if we make the substitution $s \leftrightarrow t$, $i \leftrightarrow k$ and $j \leftrightarrow l$ in (2) we obtain (3). In a similar manner, in (4) the

substitution $k \leftrightarrow l$ transforms it in (5) and the substitution $s \leftrightarrow t$, $j \leftrightarrow k$, $i \leftrightarrow l$ yields (5) $\leftrightarrow$ (6). There are several other symmetry properties of the box diagram which transform one solution into another.

Now we must combine them in such a way to have sums of linearly independent solutions bearing the same functional variable. This is the constructive prescription[1]. We then get from the above list six types of functional variables, that is, six new such combinations or six new results for the Feynman integral (1), namely,

$$J_3 = I_6, \qquad J_4 = I_7, \tag{14}$$

$$J_5 = I_8 + I_9, \qquad J_6 = I_{10} + I_{11}, \tag{15}$$

$$J_7 = I_{12} + I_{13}, \qquad J_8 = I_{14} + I_{15}. \tag{16}$$

Note that the relevant Feynman integral is obtained via

$$K(i, j, k, l; m) \equiv J(-i, -j, -k, -l; m)$$

3 Regularization and Discussion.

Here we constrain ourselves to the special case where the integral (1) is the one for QED photon-photon scattering at the one-loop level, that is, we are interested in taking the particular values $i = j = k = l = -1$. However, these expressions become singular when we take the referred limit and/or let $D = 4$. Therefore, some kind of regularization procedure is called for.

For the first two, i.e., J_3 and J_4, we can adopt the standard procedure of dimensional regularization[6]. Introduce $D = 4 - \varepsilon$ and expand the whole expression around $\varepsilon = 0$ to get

$$I_6^R = \frac{8\pi^2}{s^2}\left[\frac{-2}{\varepsilon} + \log\left(-2\pi s\right) + \gamma_E\right]F_3(1, \tfrac{1}{2} + \tfrac{1}{2}\varepsilon, 1, 1 + \tfrac{1}{2}\varepsilon; 2 + \tfrac{1}{2}\varepsilon|x, y), \tag{17}$$

where $x = -t/s$, $y = 4m^2/s$, F_3 is a hypergeometric function of two variables which is absolutely convergent for $|x| < 1$ and $|y| < 1$, and γ_E is the Euler's constant[10,12]. We can write a simpler expression by using a reduction formula [10,12,13],

$$F_3(\alpha, \alpha', \beta, \gamma - \beta; \gamma|x, y) = \frac{1}{(1-y)^{\alpha'}}F_1(\beta, \alpha, \alpha'; \gamma|x, z), \tag{18}$$

where $z = y/(y-1)$ and F_1 is another hypergeometric function of two variables which is absolutely convergent in the same region of the F_3 above. This function has a simple integral representation[12],

$$F_1(\alpha, \beta, \beta'; \gamma | z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 du \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-uz_1)^\beta (1-uz_2)^{\beta'}}, \quad (19)$$

where the parameters must satisfy $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma - \alpha) > 0$. It is straightforward to evaluate this integral when the parameters take the values we have in hands. Substituting (19) and (18) in (17) and expanding the hypergeometric function in Taylor series, we get

$$J_3(-1, -1, -1, -1; m) \equiv I_6^R = \frac{8\pi^2}{s(s - 4m^2)} \left[\frac{-2}{\varepsilon} - \partial_{\beta'} - \partial_\gamma + \log(-2\pi s) \right.$$
$$\left. + \gamma_E + \log\left(1 - \frac{4m^2}{s}\right) \right] F_1(\alpha, \beta, \beta'; \gamma | x, z). \quad (20)$$

Note that there is a simple pole which we did not expect by naive power counting. We will discuss this singularity and the one that appears in the following solution in the next subsection. Here we introduce the parametric derivatives[8,14],

$$\frac{\partial(\alpha|z)}{\partial\alpha} \equiv \partial_\alpha(\alpha|z) = (\alpha|z)\left[\psi(\alpha + z) - \psi(\alpha)\right], \quad (21)$$

where the $\psi-$function is the logarithmic derivative of the gamma function [12,15]. First carry out the parametric derivatives in (19) then substitute the values of the parameters and integrate. For the other terms the integral results in,

$$F_1(1, 1, \tfrac{1}{2}; 2 | x, z) = -\frac{s}{t} \frac{1}{R_{st}} \log\left(\frac{1 + R_{st}}{1 - R_{st}} \frac{R_s - R_{st}}{R_s + R_{st}}\right), \quad (22)$$

where

$$R_s = \sqrt{1 - \frac{4m^2}{s}}, \qquad R_{st} = \sqrt{1 - \frac{4m^2}{t} - \frac{4m^2}{s}}. \quad (23)$$

See that the limit $t \to 0$ is well-defined.

We can write down immediately the result for the integral I_7 by noting that it can be transformed into I_6 if we make the changes $i \leftrightarrow k$, $j \leftrightarrow l$ and $s \leftrightarrow t$,

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$$J_4(-1,-1,-1,-1;m) \equiv I_7^R = \frac{8\pi^2}{t(t-4m^2)} \left[\frac{-2}{\varepsilon} - \partial_{\beta'} - \partial_\gamma + \log\left(-2\pi t\right) \right.$$

$$\left. +\gamma_E + \log\left(1 - \frac{4m^2}{t}\right) \right] F_1(\alpha, \beta, \beta'; \gamma|w, w'). \, (24)$$

where $w = -s/t$ and $w' = 4m^2/(4m^2 - t)$. For the region of convergence see figure 1.

For the remaining solutions dimensional regularization is unsuitable to regularize their divergences. Consider for example (4) where there is a factor $(-i|i - k + l)$ which is divergent in the particular limit we are interested in, i.e., $i = j = k = l = -1$. This factor has no D-dependence and dimensional regularization here is useless. What we must do is to use a different procedure, namely, regularizing the exponent of some of the propagators[1,8,16].

Let us then consider the fifth solution of the Feynman integral, J_5. We must regularize one exponent of one of the propagators, say, $k = 1 - \zeta$ (we could also take the exponent l.). The important point is that the final result will be independent of this choice. The other exponents are set to minus one while the dimension of the space-time remains arbitrary. Doing this we have

$$I_8^R = (-\pi)^{D/2} \frac{1}{st^{3-D/2}} \frac{\Gamma(3 - \frac{1}{2}D + \zeta)\Gamma(\zeta)\Gamma^2(\frac{1}{2}D - 2 - \zeta)}{t^\zeta \Gamma(1 + \zeta)\Gamma(D - 4 - \zeta)}$$

$$\times H_2\left(-\zeta, \, 1, 1 + \frac{1}{2}\zeta, \, \frac{1}{2} + \frac{1}{2}\zeta, 1 - \zeta \, \middle| \, \frac{-t}{s}, \frac{-4m^2}{t}\right), \qquad (25)$$

and

$$I_9^R = (-\pi)^{D/2} \frac{\Gamma(3 - \frac{1}{2}D)\Gamma^2(\frac{1}{2}D - 2)}{st^{3-D/2}} \frac{\Gamma(-\zeta)}{s^\zeta \Gamma(D - 4 - \zeta)}$$

$$\times H_2\left(0, \, 1 + \zeta, 1 + \frac{1}{2}\zeta, \, \frac{1}{2} + \frac{1}{2}\zeta, 1 + \zeta \, \middle| \, \frac{-t}{s}, \frac{-4m^2}{t}\right). \qquad (26)$$

The hypergeometric function H_2 is defined by the double sum[12],

$$H_2(\alpha, \beta, \gamma, \delta; \rho|x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha|m - n)(\beta|m)(\gamma|n)(\delta|n)}{(\rho|m)} \frac{x^m y^n}{m!n!}. \qquad (27)$$

The region of absolute convergence of this function $H_2(\ldots|x, y)$ is bounded by the lines[12],

$$|y| < \frac{1}{1 + |x|}, \qquad |x| < 1, \qquad |y| < 1, \qquad (28)$$

see figure 2.

In proceeding our analysis of the new results, for the solution J_5, let us now expand the H_2 function in Taylor series around $\zeta = 0$, keeping terms up to the first order in ζ

$$
\begin{aligned}
J_5(-1,-1,-1,-1;m) = {} & \frac{2^{5-D}(-\pi)^{D/2}\sqrt{\pi}\,\Gamma(3-\frac{1}{2}D)\Gamma(\frac{1}{2}D-2)}{st^{3-D/2}\Gamma(\frac{1}{2}D-\frac{3}{2})} \Big[-\gamma_E \\
& + \log\left(\frac{s}{t}\right) - 2\psi\left(\tfrac{1}{2}D-1\right) + \psi\left(3-\tfrac{1}{2}D\right) + \frac{4}{D-4} - \partial_\alpha - \partial_\rho\Big] \\
& \times H_2\left(\alpha,1,1,\tfrac{1}{2};\rho \left|\frac{-t}{s},\frac{-4m^2}{t}\right.\right),
\end{aligned}
\tag{29}
$$

where the parametric derivatives must be taken at the point $\alpha = 0;\;\; \rho = 1$. The nature and meaning of these singularities will be the touched on in the following subsection.

In a similar manner we regularize the sixth solution. But now we take $i = -1-\zeta$. As a result we get,

$$
\begin{aligned}
J_6(-1,-1,-1,-1;m) = {} & \frac{2^{5-D}(-\pi)^{D/2}\sqrt{\pi}\,\Gamma(3-\frac{1}{2}D)\Gamma(\frac{1}{2}D-2)}{ts^{3-D/2}\Gamma(\frac{1}{2}D-\frac{3}{2})} \Big[-\gamma_E \\
& + \log\left(\frac{t}{s}\right) - 2\psi\left(\tfrac{1}{2}D-1\right) + \psi\left(3-\tfrac{1}{2}D\right) + \frac{4}{D-4} - \partial_\alpha - \partial_\rho\Big] \\
& \times H_2\left(\alpha,1,1,\tfrac{1}{2};\rho \left|\frac{-s}{t},\frac{-4m^2}{s}\right.\right),
\end{aligned}
\tag{30}
$$

and as we shall see later on, these two functions H_2 are related to the functions F_3 that are divergent too. The region of convergence can be constructed as we did above for the H_2 function (see fig.3).

Consider now the seventh solution of the Feynman integral, J_7. Like the preceding case, it has a simple pole in the exponents, so that it demands only one suitable parameter to regularize it. Looking at (8) and (9) we note that a good choice to introduce our regularization parameter is to take $l = -1-\zeta$, while the other exponents can be set to $i = j = k = -1$ without any problem. Then,

$$
\begin{aligned}
I_{12}^R = {} & \frac{-\pi^2}{m^2 s}\left(-\frac{1}{\zeta} - 1 + \log s + O(\zeta)\right) \\
& \times \mathcal{S}_2\left(-\tfrac{1}{2}+\tfrac{1}{2}\zeta,1+\zeta,1,1,3-\tfrac{1}{2}D;1+\zeta,1-\tfrac{1}{2}\zeta \left|\frac{4m^2}{s},\frac{-t}{4m^2}\right.\right),
\end{aligned}
\tag{31}
$$

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and

$$I_{13}^R = \frac{-\pi^2}{m^2 s} \left(\frac{1}{\zeta} - 1 - \gamma_E - \log\left(-m^2\right) + O(\zeta) \right)$$
$$\times \mathcal{S}_2 \left(-\tfrac{1}{2} - \tfrac{1}{2}\zeta, 1, 1, 1, 3 - \tfrac{1}{2}D + \zeta; 1 - \zeta, 1 + \tfrac{1}{2}\zeta \left| \frac{4m^2}{s}, \frac{-t}{4m^2} \right. \right) . \quad (32)$$

Now expand the factors of (31), (32) and the series (13) around $\zeta = 0$ and substitute the values $\alpha = -\frac{1}{2}$, $\beta = \gamma = \delta = \rho = \phi = \phi' = 1$. Using the fact that $\partial_\beta + \partial_\rho = 0$ (only because these two parameters are equal) and an analogous relation between ϕ and ϕ', we get, in four dimensions,

$$J_7(-1,-1,-1,-1;m) = \frac{\pi^2}{m^2 s} \left[2 + \gamma_E + \partial_\alpha + \partial_\rho - \log\left(\frac{-s}{m^2}\right) \right]$$
$$\times H_2 \left(\alpha, 1, 1, 1; \rho \left| \frac{4m^2}{s}, \frac{-t}{4m^2} \right. \right) . \quad (33)$$

Note that the above result is finite and that there is no dependence on ϕ and ϕ', so $\mathcal{S}_2$ reduces to H_2. The pole cancels out and then we can take the limit of vanishing ζ.

The next two solutions follow the same procedure, yielding

$$J_8(-1,-1,-1,-1;m) = \frac{\pi^2}{m^2 t} \left[2 + \gamma_E + \partial_\alpha + \partial_\rho - \log\left(\frac{-t}{m^2}\right) \right]$$
$$\times H_2 \left(\alpha, 1, 1, 1; \rho \left| \frac{4m^2}{t}, \frac{-s}{4m^2} \right. \right) , \quad (34)$$

which is also finite. The region of convergence of this solution is shown in figure 4. We do not need to calculate the parametric derivatives because Davydychev already did it[8]. Using the transformation formula between H_2 and F_2[13],

$$H_2(\alpha, \beta, \gamma, \delta; \rho | x, y) = \mathcal{A}_1 \, F_2 \left(\alpha + \gamma, \beta, \gamma; \rho, 1 + \gamma - \delta \left| x, \frac{-1}{y} \right. \right)$$
$$+ \mathcal{A}_2 \, F_2 \left(\alpha + \delta, \beta, \delta; \rho, 1 + \delta - \gamma \left| x, \frac{-1}{y} \right. \right) , \quad (35)$$

where we define the coefficients

$$\mathcal{A}_1 = \frac{\Gamma(1-\alpha)\Gamma(\delta-\gamma)}{\Gamma(\delta)\Gamma(1-\alpha-\gamma)} y^{-\gamma}, \qquad \mathcal{A}_2 = \mathcal{A}_1(\gamma \leftrightarrow \delta) \quad (36)$$

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we can identify the parametric derivatives of H_2 with the ones of F_2 calculated by Davydychev. Care must be taken with (35) because with the particular parameters we have in hands the individual terms on the RHS are singular, but which cancels out at the end when both terms are added together.

3.1 Discussion.

As we have mentioned earlier, the set of new solutions we have obtained here contains singular solutions that deserve a closer look. Let us examine them in order to understand the meaning and the nature of such singularities. To begin with, let us give some arguments to show the correctness of our results. As we had conjectured in [1], solutions containing the H_2 hypergeometric functions did in fact appear.

Consider the first result we obtained in our previous work[1,8], i.e.,

$$J_1(-1,-1,-1,-1;m) = \frac{\pi^2}{6m^4} F_3\left(1,1,1,1;\frac{5}{2}\left|\frac{s}{4m^2},\frac{t}{4m^2}\right.\right). \qquad (37)$$

The hypergeometric function F_3 which appears here is related to the hypergeometric function H_2 via analytic continuation (see Erdélyi[13]),

$$F_3(\alpha,\alpha',\beta,\beta';\gamma|x,y) = \mathcal{B}_1\ H_2\left(1+\alpha-\gamma,\alpha,\alpha',\beta';1+\alpha-\beta\left|\frac{1}{x},-y\right.\right) \qquad (38)$$
$$+\mathcal{B}_2\ H_2\left(1+\beta-\gamma,\beta,\alpha',\beta';1+\beta-\alpha\left|\frac{1}{x},-y\right.\right),$$

where the two coefficients are

$$\mathcal{B}_1 = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-x)^{-\alpha}, \qquad \mathcal{B}_2 = \mathcal{B}_1(\alpha \leftrightarrow \beta). \qquad (39)$$

So, with the help of equation (38) we rewrite F_3 in terms of H_2 without worrying very much about constant factors because they arrange themselves properly in the process. Indeed, in this case both factors on the RHS containing gamma functions are singular (this is a special case of analytic continuation known as the logarithmic case), but whose singularities cancel out at the end, leaving us with a finite result as it should be. Then,

$$J_1 \sim F_3\left(1,1,1,1;\frac{5}{2}\left|\frac{s}{4m^2},\frac{t}{4m^2}\right.\right) = C_1 H_2\left(-\frac{1}{2},1,1,1;1\left|\frac{4m^2}{s},\frac{-t}{4m^2}\right.\right)+$$

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$$+C_2 H_2 \left(-\frac{1}{2}, 1, 1, 1; 1 \left| \frac{4m^2}{s}, \frac{-t}{4m^2} \right. \right) , \qquad (40)$$

which clearly portrays the same H_2 function we have in J_7. Conclusion: NDIM provides, even if we did not know (38) *a priori*, the transformation $J_1 \to J_7$, or, in other words, the analytic continuation formula $F_3 \to H_2$. Moreover, as Erdérlyi[13] mentioned, there is a transformation similar to (38) for the variable y in F_3. This will give $J_1 \to J_8$.

In order to verify that there are branch points in the Feynman integral, we can do the following. Consider the definition of the hypergeometric function H_2 given in (27). Substituting the values of the parameters — recall that the derivatives of an analytic function are also analytic having the same region of convergence — two of them cancel out and we get

$$H_2 \left(-\tfrac{1}{2}, 1, 1, 1; 1 \,|x, y\right) = \sum_{\mu,\nu=0}^{\infty} \frac{(1|\nu)(1|\nu)}{(\frac{1}{2}|\nu)} \frac{(-y)^\nu}{\nu!} \left(-\tfrac{1}{2} - \nu|\mu\right) \frac{x^\mu}{\mu!} , \qquad (41)$$

where we have used the identity $(a|-k) = (-1)^k/(1-a|k)$. Observe that the series in μ is a hypergeometric function ${}_1F_0$[15] that can be summed. It results in the following

$$H_2 \left(-\tfrac{1}{2}, 1, 1, 1; 1 \,|x, y\right) = \sqrt{1-x} \sum_{\nu=0}^{\infty} \frac{(1|\nu)(1|\nu)}{(\frac{1}{2}|\nu)} \frac{[-y(1-x)]^\nu}{\nu!} , \qquad (42)$$

with variables x and y given in either J_7 or J_8. The remaining series in ν is a ${}_2F_1$ hypergeometric function that can be written down in terms of an elementary function and it is straightforward to show that it has branch points, see (46) below.

The same procedure can be applied to J_3. Using (38) for the hypergeometric function F_3, we get

$$J_3 \sim F_3 \left(1, 1, 1, \frac{1}{2}; 2 \left| \frac{-t}{s}, \frac{4m^2}{s} \right. \right) = C_3 H_2 \left(0, 1, 1, \frac{1}{2}; 1 \left| \frac{-s}{t}, \frac{-4m^2}{s} \right. \right) +$$

$$+ C_4 H_2 \left(0, 1, 1, \frac{1}{2}; 1 \left| \frac{-s}{t}, \frac{-4m^2}{s} \right. \right) , \qquad (43)$$

yielding $J_3 \to J_6$. The analogous transformation for the variable y in F_3 yields $J_3 \to J_5$ and so on.

Using analogous routes we used above, it is possible to express this H_2 function in terms of an elementary function, this time a square root. Considering its

definition, the canceling of the parameters β and ρ for the specified values and summing the series in ν we get,

$$H_2\left(0,1,1,\tfrac{1}{2};1\,|x,y\right) = \sum_{\nu=0}^{\infty} \frac{[-y(1-x)]^\nu (\tfrac{1}{2}|\nu)}{\nu!} = \frac{1}{\sqrt{1+y(1-x)}}\,, \qquad (44)$$

observe that the square root in the denominator is equal to R_{st}, see eq.(23). An important point to note here is that even though the results remain divergent, they are still connected by an analytic continuation formula. The questions that need to be addressed then are now: *What does this mean? What is the nature of these singularities?*

First of all, it is known[17] that a four-point graph like the one in the photon-photon scattering has no leading singularities in the physical region. Such singularities does happen to occur in four-point functions when the two incoming particles enter the same vertex and the two outgoing particles also leave the same vertex (see Fig. 5). Since this is not our case, we thus conclude that the singularities we have do not occur on the physical sheet, i.e., they are harmless[17].

Secondly, in analytically continuing a given function from a region $\mathcal{R}_1$ into another region, $\mathcal{R}_2$ it is important that no singularities be present between the regions, otherwise the result for the analytic continuation may not be unique. The non-uniqueness always manifest itself whenever the singularity is of the branch point type[18]. We know that for the photon-photon scattering process we have a branch-cut in $s = 4m^2$ in the s-channel, so that in carrying out our analytic continuation from $J_6 \to J_3$, we are crossing this branch-cut, and then the singularities do arise.

This naive argumentation shows us the great possibilities of NDIM. It reproduces three general — with no restriction in the parameters — analytic continuation relations between Appel's hypergeometric functions which are far from trivial to obtain (see [13]). It is clear too that the technique allows us the bonus by-product of pinpointing singularities of Feynman integrals.

Eden[19] devised a technique to find out the singularities of integral representations. In [17] Eden *et al* applied it in the general box diagram and the equation of Landau's surface — the surface of possible singularities of a integral representation — is given by a 4x4 determinant. In our case (equal mass for the virtual matter fields and on-shell photons) the Landau's equation[6,17,20] is,

$$\frac{st}{4m^6}\left(\frac{st}{4m^2} - s - t\right) = 0, \qquad (45)$$

so that there are four possible solutions,

$$s = 0, \quad t = 0, \quad s = \frac{4m^2 t}{t - 4m^2}, \quad t = \frac{4m^2 s}{s - 4m^2}. \tag{46}$$

Just here it is important to observe that the two last solutions make the hypergeometric function H_2 in (44) and in (42) singular. *They are branch points of the Feynman integral.* The first two are the so-called pseudo-threshold [6,17,20,21] — singularities of the Feynman integral which occur on an unphysical sheet — see also that the possible singularities of (1) are located in the region of convergence of the two above functions, J_3 and J_4. We think that the poles does not cancel because of this reason, the so-called pinch singularities. We can verify, comparing the analysis contained in [6], that the last two solutions of the Landau's equation are in fact singularities of (1).

4   Conclusion.

Using the technique known as negative dimensional integration we obtained six new results for the massive Feynman integral for the box diagram which contributes to the photon-photon scatteringin QED. These results are expressed in terms of Appel's hypergeometric functions F_3 and H_2. All these new results had to be regularized. Four of them remain divergent even after the regularization procedure, but the tenacious singularities are harmless because they do not lie in the physical region. The other two become finite after the regularization procedure and these are important in treating relativistic dynamics of forward scattering. We have shown that these new solutions are correctly related with each other by analytic continuation. To further check these results, calculation of the forward amplitude for the photon-photon scattering amplitude in QED is in progress.

Acknowledgement

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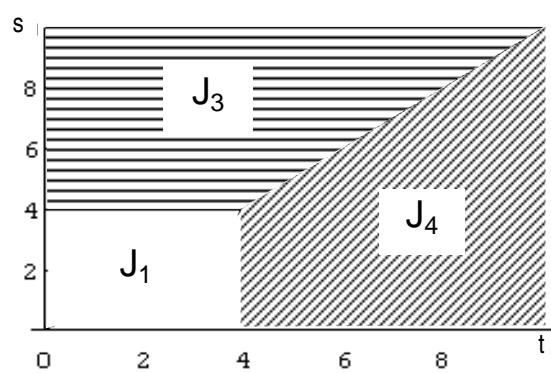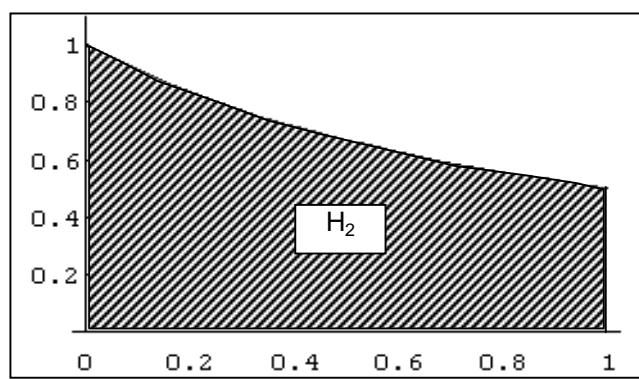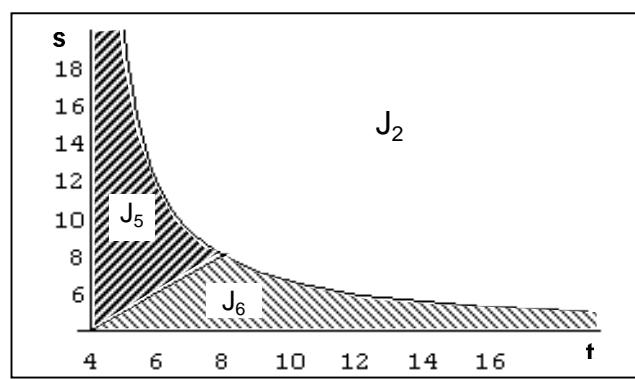
Alfredo T. Suzuki, Physics Letters B, Figure 3.